Irregular and Simulatable Functionals on Wiener Space

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We discuss the difficulties of effective statistical simulation by studying the case of functionals of multivariate Brownian motion. This leads us to a proposal of definition of s-functionals which could be a step toward a statement like a church thesis for simulatable random variates. The neologism "simulatable" is taken here in the sense of "effective for statistical simulation".

I. Different ways of simulation of Brownian functionals

In this part, we discuss the main methods for computing the expectation of Brownian functionals by simulation with a particular interest on $a.s.$ and L^p approximation.

A. Almost sure versus L^p approximation.

Let us consider for the discussion the case of a stochastic differential equation of Ito's type

$$X_t^x = x + \int_0^t \sigma(X_s^x)dBs + \int_0^t b(X_s^x)ds$$

where $x \in \mathbb{R}^n, (B_t)_{t\geq 0}$ is a $d-$dimensional standard Brownian motion starting at 0, and where $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ are suitably regular.

It is well known (see especially [16] [10] [27]) that if the Frobenius commutativity condition

$$\forall x \in \mathbb{R}^n \quad \sigma'(x)[\sigma(x)u]v = \sigma'(x)[\sigma(x)v]u \qquad \forall u, v \in \mathbb{R}^d$$

is fulfilled then a version of the solution can be chosen such that the Ito mapping

$$(x,\omega) \in \mathbb{R}^n \times C([0,1], \mathbb{R}^d) \to X_.^x(\omega) \in C([0,1], \mathbb{R}^n)$$

be locally Lipschitz. This occurs in particular when $d = 1$. In that case it is possible to construct almost sure and L^p approximations $X_.^{x,n}(\omega)$ of $X_.^x(\omega)$ by approximating the Brownian path ω and using the continuity of the Ito mapping (see e.g. [6] [28] [21]). Actually, with or without the Frobenius assumption, direct discretization schemes $X_.^{x,n}$ can be constructed which converge in L^p and for some schemes almost surely, let us quote among a large literature on this subject [17] [7] [22] [19] [29] [11].

Now there are essentially two ways of using these approximations to compute numerically the expectation of the functional $F(\omega) = f(X_.^x(\omega))$ where f is a regular functional :

1. By L^P-approximation results, a functional F_ε is first chosen, defined on a finite (eventually high) dimensional space, such that

$$\| F - F_\varepsilon \|_{L^1} \leq \varepsilon$$

and then $\mathbb{E}[F_\varepsilon]$ is computed by simulation that is to say

- either by classical Monte Carlo using the strong law of large numbers

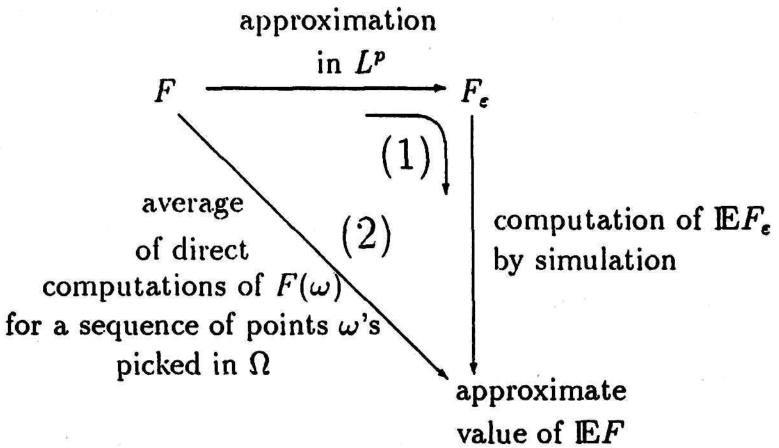
$$\mathbb{E}F_\varepsilon = \lim_{N\uparrow\infty} \frac{1}{N}\sum_{n=1}^{N} F_\varepsilon(\omega_n)$$

where the ω_n's are picked out of Ω independently,
- either by an ergodic transform using Birskhoff's theorem

$$\mathbb{E}F_\varepsilon = \lim_{N\uparrow\infty} \frac{1}{N}\sum_{n=1}^{N} F_\varepsilon(\tau^n(\omega))$$

(see for instance [4] [2])
- or using equidistributed sequences.

2. An approximate value of $F(\omega)$ is computed by taking $f(X^{x,n}(\omega))$ for n sufficiently large where $X^{x,n}$ is an almost sure approximation of the solution of the S.D.E. This procedure is repeated either for independently picked ω_n's $n \geq 1$ and then averaging or for a sequence of points $\tau^n(\omega)$ (ergodic theorem) etc.



We emphasize the following points : only the second method uses almost sure approximation results.The first one only needs L^p approximation or even approximation in law, (see part B below). In the second step of the first method, because the functional F_ε is chosen to be regular and defined on a finite dimensional space, there exists in general stopping criterion for the computation of $\mathbb{E}F_\varepsilon$ by simulation. On the contrary, in the second method such stopping criteria are not always available. Let us remark, at least, that if, in the second method, we stop at an N which is selected independently of w, we obtain actually a copy of the first method.

Let us consider a very simple example :

Let

$$F(\omega) = \frac{1}{1 + (\int_0^1 |\omega(s)|ds)^2}$$

be defined on $\mathcal{C}([0,1],\mathbb{R})$. Here F is continuous and even Lipschitz with ratio 1 on $\mathcal{C}([0,1],\mathbb{R})$ equipped with the uniform norm. Thus the first method applies without difficulty : If

$$G^n(\omega) = \sum_{k=0}^{2^n-1}\left[\omega(\frac{k}{2^n})(\frac{k+1}{2^n} - t) + \omega(\frac{k+1}{2^n})(t - \frac{k}{2^n})\right] 2^n.1_{[\frac{k}{2^n},\frac{k+1}{2^n}[}(t)$$

is the piecewise linear dyadic approximation of the order n of ω, and if we take $F_n(\omega) = F(G^n(\omega))$, we have

$$|\mathbb{E}F - \mathbb{E}F_n| \leq \| \sup_{s\in[0,1]} |\omega(s) - G^n(\omega)(s)| \|_2 \leq \frac{K}{2^{n/2}}$$

where the constant K can be explicitely computed. Now, a problem with the second method is to write down a stopping criterion which garrantees that $G^n(\omega)$ is near ω using only informations concerning $G^n(\omega)$. We shall come back to this key point in part II and part IV.

B. About the approximation in law.

Especially in the case of Wiener functionals, the approximation in law was largely studied in the literature, it is based on an invariance principle such as the Doob-Donsker theorem or on some of it generalisations to diffusion processes (see [14], [18], [23]). For a bounded continuous functional F or Riemann-integrable with respect to Wiener measure, it allows to approximate $\mathbb{E}F$ by the expectation of $F(Z.)$ where $(Z_t)_{t\leq 0}$ is a process which approximates weakly the Brownian motion. Just as there are results on the speed of convergence in the central limit theorem (cf. [25]), estimates of the speed of convergence in the approximation of $\mathbb{E}F$ by $\mathbb{E}F(Z.)$ can be obtained. Thus in the discussion of the preceding paragraph the approximation in law plays a similar role as the approximation in L^p : It yields a functional F_ϵ (on a different probability space) such that $\mathbb{E}F_\epsilon$ is near $\mathbb{E}F$ and can be computed by simulation.

Let us remark nevertheless that the weak convergence can often be improved into stronger convergences. On one hand there is a well known general theorem of Skorohod which gives a framework where it becomes a convergence in probability, but on the other hand in the concrete cases of explicit settings this transformation can be done very naturally in general. So, in finite dimension for example, where the changes of spaces are easy, one has practically never to use weak convergence for computing an expectation.

II. Non-Riemann-integrability of multiple Wiener integrals.

When the Frobenius commutativity condition fails, the solution of SDE's with regular coefficients are not regular. To make the discussion simple we take the case of

$$F(\omega) = \int_0^1 \omega_s^1 d\omega_s^2$$

where $\omega = (\omega^1, \omega^2) \in \Omega = \mathcal{C}([0,1], \mathbb{R}^2)$. (We know, see especially [?],[1], [3], that multiple Wiener integrals are generic examples of diffusions at least for SDE's with analytical coefficients). In this case, F has no continuous version (see e.g. [31] and [20] for stronger results) and it is not difficult to prove that F is discontinuous at every point of $\mathcal{C}([0,1], \mathbb{R}^2)$:

Lemma 1 . *For every open set $G \subset \mathcal{C}([0,1], \mathbb{R}^2)$ and $a \in \mathbb{R}$*

$$\mathbb{P}[\{F > a\} \cap G] > 0 \textit{ and } \mathbb{P}[\{F < a\} \cap G] > 0.$$

Proof

a) Let us first show that

$$\forall a \in \mathbb{R},\ \forall r > 0 \qquad \mathbb{P}\{\int_0^1 \omega_s^1 d\omega_s^2 > a, \| \omega \| < r\} > 0.$$

For this, by the relation

$$\omega^1(1)\omega^2(1) = \int_0^1 \omega_s^1 d\omega_s^2 + \int_0^1 \omega_s^2 d\omega_s^1,$$

it is sufficient to prove

$$\forall a \in \mathbb{R}, \forall r > 0 \qquad \mathbb{P}\{\int_0^1 \omega_s^1 d\omega_s^2 - \int_0^1 \omega_s^2 d\omega_s^1 > a, \| \omega \| < r\} > 0.$$

Let $K(t)$ be the increasing process

$$K(t) = \int_0^t [(\omega_s^1)^2 + (\omega_s^2)^2] ds$$

then $\int_0^1 \omega_s^1 d\omega_s^2 - \int_0^1 \omega_s^2 d\omega_s^1$ can be written (cf. [12]) $B(K(1))$ where $(B_t)_{t\geq 0}$ is a Brownian motion independent of $((\omega_t^1)^2 + (\omega_t^2)^2)_{t\geq 0}$. So the probability to be evaluated is equal to

$$\mathbb{E}[f(K(1))1_{\|\omega\|<r}]$$

with $f(t) = \mathbb{P}\{B(t) > a\}$ and is strictly positive since $f > 0$ and $\mathbb{P}\{\| \omega \| < r\} > 0$.

b) Let G be an open set. By density there is a Cameron-Martin function $h \in G$. By Cameron-Martin theorem under $\mathbb{P}_h$ (which is equivalent to $\mathbb{P}$) $\omega - h$ is a Brownian motion to which part a) applies, which completes the proof. □

Let us quote here the important result of Stroock and Varadhan [26] that under suitable regularity assumptions on the coefficient of the SDE, the solution can be redefined to be approximately continuous at points ω belonging to C^∞.

By the lemma if φ is a continuous non constant bounded function, the map $\omega \rightarrow \varphi(\int_0^1 \omega_s^1 d\omega_s^2)$ is not Riemann integrable. This is a fortiori still true when the Wiener space is equipped with weaker topologies such as the topology of pointwise convergence on dyadic points which we shall use later on.

Let us now emphasize that an almost sure approximation scheme is useless for direct simulation unless one has constructed a pointwise stopping criterion. Explicit examples are given in [4].

III. L^p-approximation results.

The preceding discussion shows the importance of L^p-approximation results and we quote here for completeness a result which can be used in applications. (see [15] [32] [30] [24] [11] for related results).

Consider the strong solution of the equation

$$X_t = x + \int_0^+ \sigma(X_s, s) dB_s + \int_0^+ b(X_s, s) ds$$

with

$$\sigma : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n\times d} \qquad\qquad b : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$$

and $\forall x, y \in \mathbb{R}^n$ and $s, t < T$

$$|\sigma(x,s) - \sigma(y,t)| + |b(x,s) - b(y,t)| \leq K(T)(|x - y| + |t - s|^\alpha)$$

for an $\alpha > 0$, where $|.|$ is an Euclidean norm.

Let $0 < t_1 < \ldots < t_p < t_{k+1} < \ldots < T$ be a partition of $[0,T]$ and $\tau = \sup_k |t_{k+1} - t_k|$, the Euler scheme Y_t is defined by induction on k by

$$\begin{aligned} Y_0 &= x \\ Y_t &= Y_{t_k} + \sigma(Y_{t_k}, t_k)(B_t - B_{t_k}) + b(Y_{t_k}, t_k)(t - t_k) \end{aligned}$$

for $t \in [t_k, t_{k+1}[$.

Then for $p \in [1, \infty[$, there is a constant $G = C(p, K(T))$ such that

$$\| \sup_{t\in[0,T]} |Y_t - X_t| \|_p \leq \tau^{\alpha\wedge 1/2}.$$

As pointed out by O. Faure [11] this implies that, taking a dyadic partition of $[0,T]$, the corresponding $Y^{(n)}$ approximation converges almost surely to X for the uniform norm on $[0,T]$. A similar result holds for the piecewise linear Euler Scheme Z defined by

$$Z_t = Y_{t_k}\frac{t_{k+1} - t}{t_{k+1} - t_k} + Y_{t_{k+1}}\frac{t - t^k}{t_{k+1} - t_k} \quad \text{for } t \in [t_k, t_{k+1}[$$

which allows a implementation by recursive simulation of the Brownian path. By the preceding discussion, the practical use of this fact is, however, questionable in general unless for particular functionals as we shall see below.

Remark. What makes problem in the use of almost sure approximation is not that it converges a priori only outside a negligible set (it can be shown moreover, under suitable regularity assumptions on the coefficients of the EDS, that the Euler scheme converges also in the quasi-everywhere sense, that is to say outside a set of zero capacity for the Dirichlet form associated with the Ornstein-Uhlenbeck process on Wiener space [5]). Even if the approximation would converge everywhere the irregularity of the functional puts a stopping criterion problem.

IV. Simulatable functionals.

In this part, we leave the Wiener space for more general frameworks and we draw from the preceding discussion a definition of functionals able to be pointwise simulated. We introduce first the ideas in the case of discrete probabilities.

A. Simulatable functionals on $\{0,1\}^{\mathbb{N}}$.

Let $(x_n)_{n\in\mathbb{N}}$ the coordinate mappings of $\{0,1\}^{\mathbb{N}}$ into each factor. As usual, we introduce the σ-fields $\mathcal{A}_n = \sigma(x_m\ m \leq n)$ and the product probability $\mathbb{P} = \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)^{\otimes\mathbb{N}}$ on $\mathcal{A}_\infty$.

Definition 1 . *A Random variable from $\{0,1\}^{\mathbb{N}}$ into $\mathbb{R}$ is said to be* **simple** *if there exists an $(\mathcal{A}_n)$-stopping times T $\mathbb{P}$-almost surely finite such that F be $\mathcal{A}_T$-measurable.*

Often F has to be supposed moreover bounded (cf. [4] 1.4 a)).

By the fact $F = \sum_{n=0}^{\infty} F.1_{\{T=n\}} p.s.$, the law of F is necessarily discrete :

$$\mu = \sum_{k=1}^{\infty} P_k\delta_{a_k}.$$

It can be shown that an $\mathcal{A}_n$-stopping times S and a $(\mathcal{A}_S)$-measurable random variable G with law μ can be constructed such that $\mathbb{E}S$ be minimal. For $x \in [0,1]$, let $\{x\}$ be the fractional part of x, let us put $\nu(x) = \sum_{n=0}^{\infty} \frac{\{2^n x\}}{2^n}$. Then the minimal value of $\mathbb{E}S$ (finite or infinite) is (cf. [13])

$$\sum_{k=1}^{\infty} \nu(p_k).$$

In general however, the laws of the simple random variables to be simulated, *are not known*. It is precisely for this reason that the simulation is usefull. Examples are numerous : - solving the Dirichlet problem in an open set of $\mathbb{R}^d$ for large d by spatial discretization and simulation of a symmetric random walk - pricing of European or American options by discretization of the Black-Scholes model, cf. [8] - modelling of queuing systems, etc.

B. Simulatable functionals on $[0,1]^{\mathbf{N}}$.

We adopt now the hypothesis that it is possible to pick out of $[0,1]$ a sequence of points according to Lebesgue measure and independently for reasoning on the simulation algorithms as does for example L. Devroye in his book [9].

Let $(U_n)_{n\geq 0}$, be the coordinate mappings of $[0,1]^{\mathbf{N}}$, $F_n = \sigma(U_m, m \leq n)$, $\mathbb{P} = dx^{\otimes \mathbf{N}}$. We are going to consider random variables F defined on $[0,1]^{\mathbf{N}}$ and F_T-measurable for some (F_n)-stopping time T $\mathbb{P}$-almost surely finite. *But which regularity has to be asked on F to get an interesting definition ?*

At least, F has to be supposed $\mathbb{P}$-Riemann integrable (with $[0,1]^{\mathbf{N}}$ equipped with the product topology) such that, outside a negligible set, the value of F at a point x can be approximated by an approximate knowledge of x. However, it is not difficult to define, quite explicitely a Borelian mapping χ from $[0,1]$ into $[0,1]^{\mathbf{N}}$ such that χ be continuous outside a negligible set and that the image of the measure dx be $dx^{\otimes \mathbf{N}}$: If

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \qquad x_k \in \{0,1\}$$

is the canonical dyadic expansion of $x \in [0,1]$, define

$$y = \chi(x) \quad \text{to be} \quad y = (y_n)_{n\geq 0} \text{ with}$$

$$y_n = \sum_{k=0}^{\infty} \frac{x_{k+\frac{(n+k)(n+k+1)}{2}}}{2^{k+1}}.$$

Thus, if F is only supposed to be $\mathbb{P}$-Riemann integrable, it is possible to come back to $[0,1]$ by considering $F o \chi$. But, by the serious fact that the digits of y_n depend on too faraway digits of x, this is useless practically for the simulation of functionals of random processes[1].

So, we have to ask more regularity on F than the only Riemann-integrability. The following definition seems reasonable to be taken in consideration :

Definition 2 *A random variable F from $[0,1]^{\mathbf{N}}$ into $\mathbb{R}^d$ is said to be* **simple** *if there is an $(\mathcal{F}_n)$-stopping time T $dx^{\mathbf{N}}$-almost surely finite such that the sets $\{T = n\}$ have negligible boundaries, F is $\mathcal{F}_T$-measurable and, for every n, F is continuous in the inside of $\{T = n\}$.*

As before, F has often to be supposed also bounded. It follows from this definition that F is continuous on a set of measure 1 and is therefore Riemann-integrable if it is bounded.

[1]The mapping χ and its reciprocal are the transformations by fusion and splitting of Paley and Wiener [?] and used in the probabilistic interpretation of quantum mechanics of Wiener and Siegel [?] [?]. The physical meaning of the probabilistic hidden variable of this model was criticized for the same reason as here (see [?] p 149).

Example. Let X be a random variable with values in $\mathbb{R}^d$ with density f continuous (or Riemann integrable) on $\mathbb{R}^d$. If X is simulated by the rejection method with a random variable Y with continuous density g such that $f \leq kg$, and if Y is simulated by $Y = \varphi(x)$ $x \in [0,1]$ with φ continuous, this can be writen

$$\begin{aligned} T &= \inf\{2n+1 : kU_{2n+1} g o \varphi(U_{2n}) \leq f o \varphi(U_{2n})\} \\ \text{and} \quad X &= \varphi(U_{T-1}). \end{aligned}$$

T is an $(\mathcal{F}_n)$-stopping time with geometric law, the sets $\{T = n\}$ have negligible boundaries and X is *simple* in the sense of definition 2.

More generally

Definition 3 . *Given a probability measure μ on $(\mathbb{R}^{\mathbf{N}}, \mathcal{B}(\mathbb{R}^{\mathbf{N}}))$ and a Polish space $(E, \mathcal{E})$, we shall say that a measurable mapping G from $(\mathbb{R}^{\mathbf{N}}, \mathcal{B}\mathbb{R}^{\mathbf{N}})$ into $(E, \mathcal{E})$ is* **simple** *if there is a stopping time S for the filtration $(\mathcal{G}_n)$ of the coordinates (X_n) of $\mathbb{R}^{\mathbf{N}}$, μ-almost surely finite, such that the boundaries of the sets $\{S = n\}$ be negligible and such that G be $\mathcal{G}_n$-measurable and continuous on the inside of $\{T = n\}$ for every n.*

C. Regularity properties of simulatable functionals.

If the stopping time used in the definition is not only almost surely finite but everywhere finite, the laws of simple random variables have the property to depend continuously on the perturbations of the random number generator and analytically if the stopping time possesses an exponential moment. Let us take first the case of $\{0,1\}^{\mathbf{N}}$ and suppose that the generator yields independent bits but with law $p\delta_1 + (1-p)\delta_0$ for $p \in [0,1]$:

Proposition 1 . *1) Let F be simple on $\{0,1\}^{\mathbf{N}}$ with respect to the stopping time T which is finite, then the distribution function $H_p(t) = \mathbb{P}_p\{F \leq t\}$ under probability $\mathbb{P}_p = (p\delta_1 + (1-p)\delta_0)^{\otimes \mathbf{N}}$ is continuous in $p \in [0,1]$.*

2) If, moreover, there is an $\alpha > 0$ such that

$$\mathbb{E}_{\frac{1}{2}}[(1+2\alpha)^T] < +\infty$$

then $H_p(t)$ is analytical in p in an open set of the complex plane containing $]\frac{1}{2} - \alpha, \frac{1}{2} + \alpha[$.

Proof

1) With the above notation, one has

$$H_p(t) = \sum_{n=0}^{\infty} \mathbb{P}_p[\{F \leq t\} \cap \{T = n\}].$$

$\mathbb{P}_p[\{F \leq t\} \cap \{T = n\}]$ is a polynomial (with degree $\leq n+1$) which is positive on $[0,1]$. So, $p \to H_p(t)$ is l.s.c. and by the same argument so is $p \to 1 - H_p(t) = \mathbb{P}_p[F > t]$ hence $p \to H_p(t)$ is continuous.

2) Let $M_n = \Pi_{k=0}^{n}[1 + (2p-1)(2x_k - 1)]$ be the martingale which is the density of $\mathbb{P}_p$ with respect to $\mathbb{P}_{\frac{1}{2}}$ on $\mathcal{A}_n$.

One has

$$H_p(t) = \sum_{n=0}^{\infty} \mathbb{E}_{\frac{1}{2}}[M_n 1_{\{F \leq t\}} 1_{\{t=n\}}].$$

Let us put $q = p - \frac{1}{2}$, the expectation

$$\mathbb{E}_{\frac{1}{2}}\left[M_n 1_{\{F\leq t\}} 1_{\{T=n\}}\right] = \mathbb{E}_{\frac{1}{2}}\left[\prod_{k=0}^{n}(1+2q(2x_k-1))1_{\{F\leq t\}}1_{\{T=n\}}\right]$$

is bounded in absolute value by $(1+2|q|)^{n+1}\mathbb{P}_{\frac{1}{2}}[\{T=n\}]$.

$H_p(t)$ is therefore the sum of a series of holomorphic functions in $\{z : |z-\frac{1}{2}| < \alpha\}$ uniformly convergent in every compact of this domain and the result follows. □

These properties extend to $\mathcal{F}_T$-measurable functionals an in particular to simple functionals on $[0,1]^{\mathbf{N}}$: let h be a bounded Borelian function on $[0,1]$ such that $\int_0^1 h(x)dx = 0$, and let us put $C = \| h \|_\infty$. Suppose that our generator yield independent points distributed according to the law $(1+\lambda h(x)).dx$ for $\lambda \in [-\frac{1}{C}, \frac{1}{C}]$. Then

Proposition 2 . *1) Let F be an $\mathcal{F}_T$-measurable function from $[0,1]^{\mathbf{N}}$ into $\mathbb{R}$ where T is an $(\mathcal{F}_n)$-stopping time which is almost surely finite for all the probability measures $\mathbb{P}_\lambda = [(1+\lambda.h(x))dx]^{\otimes\mathbf{N}}$ for $\lambda \in [-a,a]$ $(a \leq \frac{1}{C})$ then the distribution function $H_\lambda(t) = \mathbb{P}_\lambda\{F \leq t\}$ is continuous in λ on $[-a,a]$.*

2) Suppose moreover $\mathbb{E}_0[(1+\alpha c)^T] < +\infty$ for some $\alpha > 0$ then $H_\lambda(t)$ is analytical in $\{\lambda : |\lambda| < \alpha\}$.

The proof is similar to preceding one.

D. Simulatable functionals on a general probability space

We are going to propose a definition of simulatable functionals in general.

Let us consider a Polish space W equipped with its Borelian σ-field $\mathcal{W}$ and with a probability m. For example $(W, \mathcal{W}, m)$ could be the Wiener space or a canonical space of some stochastic process.

As the preceding discussion shows, the topology on the space W is in general too strong to be used in simulation. Indeed to be effectively used the topology has to be such that a fundamental system of neighbourhoods of a point be composed of sets described by a finite number of rational numbers i.e. by an integer. On the Wiener space, for instance, it could be the simple convergence on rational numbers (cf. part V below).

Then we define a *presentation* of $(W, \mathcal{W}, m)$ as a copy of its measurable structure equipped with a weaker topology :

Definition 4 . *Let $(W, \mathcal{W}, m)$ be a polish space, a* **presentation** *of $(W, \mathcal{W}, m)$ in a mapping φ of W in $\mathbb{R}^{\mathbf{N}}$ such that*

- φ is an isomorphism of measurable spaces between $(W, \mathcal{W}, m)$ and $(\mathbb{R}^{\mathbf{N}}, \mathcal{B}(\mathbb{R}^{\mathbf{N}}), \mu)$ [i.e. $\mu = \varphi_* m$ and $\exists N, m(N) = 0$, $\exists N^1, \mu(N^1) = 0$ such that $\varphi : W \backslash N \to \mathbb{R}^{\mathbf{N}} \backslash N^1$ be one to one and bimeasurable],

- φ is continuous.

Then we need to be able to simulate *progressively* the coordinate process $X = (X_n)_{n\geq 0}$ of $(\mathbb{R}^{\mathbf{N}}, \mathcal{B}(\mathbb{R}^{\mathbf{N}}), \mu)$ so that if we are given a *simple* functional G on $(\mathbb{R}^{\mathbf{N}}, \mathcal{B}(\mathbb{R}^{\mathbf{N}}), \mu)$ we can simulate G too.

Definition 5 . *Let $X = (X_n)_{n\geq 0}$ the coordinate of $\mathbb{R}^{\mathbf{N}}$ and $(G_n) = \sigma(X_m\ m \leq n)$, we shall say that $\psi : [0,1]^{\mathbf{N}} \to \mathbb{R}^{\mathbf{N}}$ is a* **progressive simulation** *of $X = (X_n)_{n\geq 0}$ under μ if*

- ψ is measurable from $([0,1]^{\mathbf{N}}, \mathcal{B}[0,1]^{\mathbf{N}})$ into $(\mathbb{R}^{\mathbf{N}}, \mathcal{B}(|\mathbb{R}^{\mathbf{N}}|)$ and $\psi_*(dx^{\mathbf{N}}) = \mu$

- There exists an increasing sequence of $(\mathcal{F}_n)$-stopping times $T_k, k \geq 0$, $dx^{\otimes\mathbf{N}}$-almost surely finite such that $\forall k$ the map $(X_0, \ldots, X_k)o\psi$ from $[0,1]^{\mathbf{N}}$ into $\mathbb{R}^{k+1}$ be $\mathcal{F}_{T_k}$-measurable.

At last, we shall say that a functional F defined on $(W, \mathcal{W}, m)$ with values in the Polish space $(E, \mathcal{E})$ is an *s-functional* if a *presentation* φ of $(W, \mathcal{W}, m)$, a *simple* functional G from $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \varphi_* m)$ into $(E, \mathcal{E})$ and a *progressive simulation* ψ were found such that $F = G o \varphi$.

$$
\begin{tikzcd}
(W, \mathcal{W}, m) \arrow[dr, "\varphi"] \arrow[drr, "F"] & & \\
& (\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \varphi_* m) \arrow[r, "G"] & (E, \mathcal{E}) \\
([0,1]^{\mathbb{N}}, \mathcal{B}([0,1]^{\mathbb{N}}), dx^{\mathbb{N}}) \arrow[ur, "\psi"] & &
\end{tikzcd}
$$

The preceding definitions will be enlightened by their application to the case of Wiener space.

V. The case of Wiener space

Let $(W, \mathcal{W}, m)$ be the space of standard Brownian motion vanishing at 0, where $W = \mathcal{C}_0([0,1], \mathbb{R}^d)$, $\mathcal{W}$ is the Borelian σ-field on W equipped with the uniform convergence, m is the Wiener measure.

We shall look at several *presentations* of the Wiener space.

A. Dyadic presentation

Let, as before, X_n be the coordinate mappings of $\mathbb{R}^{\mathbb{N}}$ let us put

$$
\begin{aligned}
X_0 o \varphi(w) &= w(1) \\
X_{2^n+k} o \varphi(w) &= w\left(\tfrac{2k+1}{2^{n+1}}\right) \quad k = 0, 1, \ldots, 2^n - 1; n \geq 0.
\end{aligned}
$$

This defines a continuous mapping φ from W into $\mathbb{R}^{\mathbb{N}}$ (equipped with the product topology). Under the probability measure $\varphi_* m$ the coordinates (X_n) constitute a Gaussian process and the law of the vector $(X_0, \ldots, X_{2^n-1})$ is a permutation of the one of $(w(\frac{1}{2}), \ldots, w(\frac{k}{2^n}), \ldots, w(1))$. Clearly φ is an isomorphism of measurable spaces.

It is easy here to explicit a *progressive simulation* ψ of the process $X = (X_n)$ on the space $([0,1]^{\mathbb{N}}, \mathcal{B}[0,1]^{\mathbb{N}}, dx^{\mathbb{N}})$. This consists of simulating $X_{2^n}, X_{2^n+1}, \ldots, X_{2^{n+1}-1}$ given $X_0, \ldots, X_{2^n-1}$, and that can be done by the celebrated recursive definition of the Brownian motion of Paul Lévy. We dont write the details.

Thus we have the diagram

$$\begin{tikzcd}
(W,\mathcal{W},m) \arrow[dr, "\varphi"'] \arrow[drr, "F"] & & \\
& (\mathbb{R}^{\mathbf{N}},\mathcal{B}(\mathbb{R}^{\mathbf{N}}),\varphi_* m) \arrow[r, "G"] & (E,\mathcal{E}) \\
([0,1]^{\mathbf{N}},\mathcal{B}([0,1]^{\mathbf{N}}),dx^{\mathbf{N}}) \arrow[ur, "\psi"] & &
\end{tikzcd}$$

We shall see now that there are *simple* functionals G which define interesting s-functionals $F = Go\varphi$.

Example 1.

For $\varepsilon > 0$ let us define

$$T_\varepsilon = \inf\{2^n : \sup_{1\le k\le 2^n} |w(\frac{k}{2^n}) - w(\frac{k-1}{2^n})| \le \varepsilon\}$$

and let us denote by $L_{2^n}(w,t)$ the piecewise linear dyadic approximation of order n of $w \in W$. So, $w \to L_{2^n}(w,t)$ is an mapping with values in $\mathcal{C}_0([0,1],\mathbb{R}^d)$ equipped with its Borelian sets. This is a Polish space.

Let us put $F_\varepsilon = L_{T_\varepsilon}$.

Then F_ε is an s-functional : it factorizes in $F_\varepsilon = G_\varepsilon o\varphi$ and we have

Lemma 2 . *G_ε is a* **simple** *functional (definition 3).*

Proof. It is easily seen that we have to proved the properties of definition 3 for F_ε and T_ε when W is equipped with the topology of simple convergence on dyadic points and with the σ-fields $\mathcal{H}_{2^n} = \sigma\left(w(\frac{k}{2^n})\ k = 1,\dots,2^n\right)$.

a) Clearly T_ε is an $(\mathcal{H}_{2^n})$-stopping time m-almost surely finite, and F_ε is $\mathcal{H}_{T_\varepsilon}$-measurable.

b) Let us put $\Delta_{2^n}(W) = \sup_{1\le k\le 2^n} |\omega(\frac{k}{2^n}) - \omega(\frac{k-1}{2^n})|$.

Let us remark that the set

$$A_\varepsilon = \{w : \Delta_{T_\varepsilon}(w) < \varepsilon\}$$

is open for the topology of simple convergence on dyadic points. Indeed if $w_0 \in A_\varepsilon$ and if the $T_\varepsilon(w_0)$ points

$$(w(\frac{k}{T_\varepsilon(w_0)}),\ k = 1,\dots,T_\varepsilon(w_0))$$

are sufficiently closed respectively to the points

$$(w_0(\frac{k}{T_\varepsilon(w_0)}), k = 1,\dots,T_\varepsilon(w_0))$$

then $T_\varepsilon(w) = T_\varepsilon(w_0)$ and hence $L_{T_\varepsilon}(w)$ is closed to $L_{T_\varepsilon}(w_0)$. It follows that F_ε is continuous on the inside of $\{T_\varepsilon = 2^n\}$ and it can be proved that the boundary of $\{T_\varepsilon = 2^n\}$ is negligible. $\square$

In the same way the following functionals

$$F_\varepsilon^1 = L_{T_\varepsilon^1} \quad \text{where} \quad T_\varepsilon^1 = \inf\{2^n : \sum_k |w(\frac{k}{2^n}) - w(\frac{k-1}{2^n})| \geq \frac{1}{\varepsilon}\}$$

$$F_\varepsilon^2 = L_{T_\varepsilon^2} \quad \text{where} \quad T_\varepsilon^2 = \inf\{2^n : \sum_k |w(\frac{k}{2^n}) - w(\frac{k-1}{2^n})|^2 \in [1-\varepsilon, 1+\varepsilon]\}$$

are s-functionals.

Example 2. Let us take again the example of part II

$$F(w) = \int_0^1 \omega_s^1 d\omega_s^2.$$

Let us notice the following facts which are simple to prove.

$$i) \quad \mathbb{E}[\int_s^t w_u^1 dw_u^2 | w_s = (a_1, b_1), w_t = (a_2, b_2)] = \tfrac{a_1+a_2}{2}(b_2 - b_1)$$

$$ii) \quad \mathbb{E}[(\int_s^t w_u^1 dw_u^2 - \tfrac{a_1+a_2}{2}(b_2 - b_1))^2 | w_s = (a_1, b_1), w_t = (a_2, b_2)] \\ = \tfrac{t-s}{12}((a_2 - a_1)^2 + (b_2 - b_1)^2) + \tfrac{(t-s)^2}{6}$$

Let us put

$$F_n = \mathbb{E}[F|\mathcal{H}_{2^n}] = \sum_k \frac{w^1_{\frac{k}{2^n}} + w^1_{\frac{k+1}{2^n}}}{2} \left(w^2_{\frac{k+1}{2^n}} - w^2_{\frac{k}{2^n}} \right)$$

of course $F_n \to F$ almost surely and in L^p $p \in [1, \infty[$.

$$iii) \quad \mathbb{E}[(F - F_n)^2 | \mathcal{H}_{2^n}] = \frac{1}{12} \frac{1}{2^n} \sum_{k=0}^{2^n-1} \left| w(\frac{k+1}{2^n}) - w(\frac{k}{2^n}) \right|^2 + \frac{1}{6} \frac{1}{2^n}$$

Then putting V_n^2 for this last expression, let us define

$$S_\varepsilon = \inf\{n \geq 0 : V_n^2 \leq \varepsilon^2\}.$$

Then by the same argument as in the preceding example F_{S_ε} is an s-functional. There obviously holds $\mathbb{E}[(F - F_{S_\varepsilon})^2 | \mathcal{H}_{2^{S_\varepsilon}}] \leq \varepsilon^2$ hence taking the expectation

$$\| F - F_{S_\varepsilon} \| \leq \varepsilon.$$

Then the simulation of $h(F_\varepsilon)$ to compute the expectation of $h(F)$, for h Lipschitz and bounded say, is a manner to give an effective sense to the second method exposed in the part I.

Remark. If, instead of $\int_0^1 \omega_s^1 d\omega_s^2$ we would have taken the Lévy area itself $L(t) = \frac{1}{2}\int_0^t (\omega_s^1 d\omega_s^2 - \omega_s^2 d\omega_s^1)$ it would be easy to find explicitely a continuous random variable with the same law as $L(t)$ on the Wiener space of three dimensional Brownian motion. This is due to the independence property used in the proof of the lemma of part II. Thus the random variable $L(t)$ can be reduced in the sense of [4] section 1.3.

B. Presentation with general partitions

Instead of choosing dyadic partitions, it is possible to choose all the rational numbers. Let $r : \mathbb{N} \to \mathbf{Q} \cap [0, 1]$ be a numbering of the rational numbers $\frac{p}{q} \in [0, 1]$ and let us put

$$X_n o \varphi(w) = w(r(n))$$

with this presentation it is possible to define s-functionals which depend on the points of a partition of $[0,1]$ whose "deepth" is random as precedingly but now whose "thiness" can vary from place to place in $[0,1]$ in function of the values at points already picked. This family includes the discretization in space for diffusion studied by O. Faure in this volume. Of course a lot of other presentations of Wiener space can be constructed giving each time its own family of s-functionals.

I am gratefull to W. Kendall for comments and very usefull discussion on the concept of effectiveness in simulation. In the oral lecture, s-functionals were called **graphic functionals**. It is indeed clear now, that if one consider a graphic simulation of, e.g. a Brownian path as on the picture below, this simulation, that is to say **the program** which yields the picture, defines a map from the Wiener space into the Polish space of piecewise affine lines of the plane which is an s-functional.

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